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Quantum symmetry and fundamental S-matrix for vector perturbed WD_n models

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Abstract. Perturbations of WD_n and W_3 conformal theories which generalize the $(1, 2)$ perturbations of conformal minimal models are shown to be integrable by 'counting arguments'. The $A^{(2)}_{2n-1,q}$ and $D^{(3)}_{4,q}$ symmetries of corresponding S-matrices are conjectured and proved by explicit construction of conserved non-local charges in the WD_3 case with the proper quantum symmetry group. The gradation change of the known R-matrix with $A^{(2)}_5$ symmetry from homogeneous to spin, which turns out to be relevant for the perturbation considered, is shown to make the R-matrix crossing invariant and fixes the effective coupling constant as a function of the initial one. Using these results the fundamental S-matrix for vector perturbed minimal WD_3 models is constructed on the basis of the RSOS model with the corresponding symmetry. The S-matrix is checked to reproduce the known results in two particular cases of minimal model number: $k = 1$ and $k \to \infty$.

1. Introduction

In recent years, many massive integrable perturbations of Virasoro minimal models were found and S-matrices for them were constructed. The spectrum of such perturbations seems to be richer and the S-matrices much more complicated for the perturbations of conformal field theories (CFTs) with additional affine symmetries. Most such theories may be expressed as coset constructions of some Kac–Moody algebras at certain levels. Among these theories are adjoint perturbations of W-invariant theories [11] built on A_n series of Lie algebras (see, for example, [5, 12]) and for other series [13].

The study of integrable perturbations of Virasoro minimal models showed [1] that the structure of the $(1, 2)$ perturbation of minimal CFT, corresponding at the classical level to the Zhiber–Mikhailov–Shabat model, and its factorized scattering theory (FST), is more complicated then the structure of the $(1, 3)$ perturbation, which classically corresponds to the Sine–Gordon model. After the work of Fateev and Zamolodchikov [3], it was not clear whether there existed some general description of FST and a symmetry of the $(1, 2)$ perturbations of minimal models, because the FST symmetry group discovered turned out to be very different: E_8, E_7 and E_6 for the $p = 3$, 4 and 6 minimal models, respectively. Nevertheless, the general solution with $A^{(2)}_{2q}$ FST symmetry of any $(1, 2)$ perturbed minimal model was found by Smirnov [2], which reproduces the previously known solutions as particular cases; however, this reproduction is essentially non-trivial and is based on the properties of the representations of $sl(2)_q$ at special values of q equal to the root of unity. More detailed investigation of the $(1, 2)$ perturbation was done in [22] and [23].

Along with integrable perturbations of conformal models with Virasoro and Kac–Moody algebras, perturbations of CFT with other additional symmetries and their FST have also been studied. In [4] a few integrable perturbations of the CFT Z_N parafermionic models were studied, which are the lowest minimal models of W-invariant theories. The $(1,3)$ integrable perturbation of CFT was naturally generalized to W-invariant theories as a perturbation by the field corresponding to the adjoint representation of the algebra A_n, and their FST was constructed in [11]. In [4] and [11] the vacuum structure of the theory was conjectured to be in correspondence to the admissibility diagram of some interaction-round-the-face (IRF) models, and in [11] the S-matrix of the model was explicitly constructed by solving the Yang–Baxter equation with the use of A_n invariant IRF Boltzmann weights. The integrability of such a perturbation of an adjoint type for WX-invariant theories constructed on an arbitrary Lie group X (treated as the coset construction $X_p \times X_1/X_{p+1}$, see, for example, [5]) has been known for a long time, and some examples of corresponding FST for these models (B_n, C_n, D_n and a few others) were discussed, for example, in few recent works of Gepner [15]. At the classical level the trivial reason for integrability of the adjoint-type perturbation is expressed by the fact of correspondence of ϕ_{adj} to the maximal positive root of the algebra such that together with screening operators of the WX-invariant theory they form X-invariant affine Toda field theory (ATFT) with imaginary coupling constant [16]. In this sense the $(1,2)$ perturbation of the Virasoro minimal models may be called a vector-type perturbation (with respect to $sl(2)$).

The natural question arises: are there integrable perturbations of CFT to W-invariant theories? Recently the existence of such perturbations was pointed out [17]. It was conjectured that, $WD_n^{(p)} + \phi_{\mathrm{vect}}$ for $n \geqslant 3$ is integrable, where ϕ_{vect} is the primary field corresponding to the fundamental weight of vector representation of D_n (the field $(211\ldots1|11\ldots1)$ in the notation of [5]). The hint for the integrability of such perturbed model is actually seen even at the 'classical' level, since the perturbing field, together with the screening operators of the WD_n, give rise to the B_n imaginary coupled ATFT. It was conjectured that the FST of this integrable model should have $A^{(2)}_{2n-1,q}$ symmetry. For $n=2$ it was found that the corresponding analogue of this integrable vector perturbation is $WA_2^{(p)} + (21|11)$, and the symmetry of the corresponding FST was conjectured to be $D_4^{(3)}$. In this case the Hamiltonian of the perturbed model completed by the screening operators of WA_2 (or W_3 in the usual notation) forms the G_2 imaginary coupled ATFT.

Checks of integrability of W_3 and WD_3, which were done in [17], are exact for irrational values of central charge, since using the counting of arguments they did not take into account the summation over the root lattice in the formula of W characters, ignoring the highest null vectors. The suggestion of the symmetry of the FST theory in this work was done analysing the $p \to \infty$ and is in complete correspondence with the non-simply-laced duality of real coupled ATFT observed recently in [6, 18].

Here we will study the above-described vector perturbations of WD_n (and W_3) theories in more detail, showing explicitly the presence of $A^{(2)}_{2n-1,q}$ symmetry with the help of the non-local currents of the model, and then discuss the S-matrix construction on the basis of the $A^{(2)}_{2n-1,q}$-symmetric R-matrix built in [19]. In section 2 we start from a more rigorous check of integrability by the counting of arguments with exact calculation of the W-characters at rational values of central charge (minimal models). In section 3 non-local charges for n=3 case with the algebra $A^{(2)}_{5,q}$ are constructed explicitly, and a possible solitonic representation for it is presented. In section 4 the role of the gradation chosen for the R-matrix (as a starting point for construction of the S-matrix), which commutes with the comultiplication thus obtained, is discussed. It is shown that the spin gradation, in which the non-local charges naturally arise, makes the R-matrix crossing invariant. Moreover, it turns out that gradation transformation fixes the effective coupling constant as a function of the initial one.

In section 5 we construct the S-matrix on the basis of the IRF Boltzmann weights found recently in [10] and check the S-matrix thus obtained for some particular cases where we expect its form to coincide with some known S-matrices.

2. Vector perturbations and their integrability

Before starting with the Hamiltonian of the vector perturbation for the W_3 minimal models in the free field representation, we recall the standard notation of primary fields [6] for the $WX_n^{(p)}$ minimal model. The primary field $\Phi_{(l|l')}$ is characterized by the set of integers $(l_1, \ldots, l_r|l'_1, \ldots, l'_r)$, where r is the rank of X_n. It can be written as

$$\Phi_{(l|l')} =: \mathrm{e}^{\mathrm{i}\beta\phi(z)} : \qquad \beta = \sum_{i=1}^{r}(\alpha_+(1-l_i) + \alpha_-(1-l'_i))\omega_i \tag{1}$$

where $\alpha_+ = \sqrt{p/(p+1)}$, $\alpha_- = -\sqrt{(p+1)/p}$, and ω_i are the fundamental weights of the algebra X_n. The screening fields are : $\mathrm{e}^{\mathrm{i}\alpha_\pm\alpha_i\phi(z)}$, where α_i, $i = 1, \ldots, r$, are positive roots of X_n.

Consider the Hamiltonian for the perturbation of W_3 conformal theory by the operator $\Phi_{(21|11)}$ which has the dimension $\frac{1}{3}(1-4/(p+1))$ in the $(p, p+1)$ unitary minimal model. This Hamiltonian may be written as

$$H = \lambda \int \mathrm{d}^2z \; : (\mathrm{e}^{\mathrm{i}\alpha_+\alpha_1\phi(z)} + \mathrm{e}^{\mathrm{i}\alpha_+\alpha_2\phi(z)} + \mathrm{e}^{-\mathrm{i}\alpha_+\omega_1\phi(z)}) : \; . \tag{2}$$

The set of vectors $(\alpha_1, \alpha_2, -\omega_1)$ expressed, for example, in the standard orthonormal basis $(e_1 - e_2, e_2 - e_3, \frac{2}{3}e_1 - \frac{1}{3}e_2 - \frac{1}{3}e_3)$, obviously forms the set of roots of the G_2 affine algebra. This fact allows us to consider the above perturbed CFT as a good candidate for the integrable model, which is the G_2 ATFT with an imaginary coupling constant, and its conserved currents should have spins equal to the exponents of G_2 (3, 5) modulo its Coxeter number (6). But we should be convinced that the integrability also survives at the quantum level, and the simplest way to check it is by the counting of arguments. The dimensions of the Verma module can be extracted from the characters of the highest-weight representations of the corresponding W-algebra. Using the character formula (see, for example, [5]) for the 'completely degenerate' representations of the W-algebra

$$\chi(\Omega, \Omega') = \left[q(1/24)\prod_{i=1}^{\infty}(1-q^i)\right]^{-r} \sum_{\hat{s}\in w}\sum_{\lambda\in\Gamma_\alpha} \det(\hat{s})q^{[p\hat{s}\Omega-(p+1)\Omega'+p(p+1)\lambda]^2/2p(p+1)} \tag{3}$$

where $(\Omega, \Omega') = (\omega_i l^i, \omega_i l'^i)$ is the primary field in the notation of (1), the sums run over the elements $\hat{s}$ of the Weyl group w and the root lattice Γ_α of the Lie algebra of the rank r, and p is the number of minimal model, we found (with the help of *Mathematica*) that according to their Virasoro levels $n = 2, \ldots, 10$ the dimensions of the Verma module of the perturbing field (21|11) modulo the total derivatives are (1, 1, 2, 2, 3, 4, 6, 6, 10) for $p = 4$, (1, 1, 3, 3, 6, 7, 13, 15, 25) for $p = 5$, and (1, 1, 3, 3, 6, 7, 13, 15, 26) for $p \geqslant 6$. The same dimensions calculated for the unity operator ((11|11)-field) are (1, 1, 1, 1, 3, 1, 4, 4, 6) for $p = 4$ and (1, 1, 1, 1, 4, 2, 7, 7, 12) for $p \geqslant 5$. Comparing the dimensions of levels which correspond to spin 5, we see that there is a conserved current of that spin, as it should be according to our observation on the G_2 ATFT structure of the perturbed theory (5 is one of the exponents of $G_2^{(1)}$).

In the same way if we perturb the $WD_n^{(p)}$ minimal theory with the central charge

$$c = n\left(1 - \frac{(2n-2)(2n-1)}{p(p+1)}\right) \tag{4}$$

by the operator $\Phi_{(21\ldots1|11\ldots1)}$ with the conformal dimension

$$\Delta = \frac{k}{2(2n-1+k)} \tag{5}$$

it easily can be seen that the set of screening vertex operators, together with the perturbing one, form the potential of the imaginary coupled B_n ATFT.

Let us highlight two remarkable facts here, which will be used and discussed later. In the limit $k \to \infty$ the conformal dimension of the perturbation is going to $\frac{1}{2}$. Another feature is that for each n the central charge of the lowest minimal model $k = 1$ is equal to 1.

The check of counting arguments in this case by use of equation (3) gives the following sequences of dimensions of the Verma module at different Virasoro levels (spins) $2, \ldots, 11$: (i) $n = 3$ $(1, 1, 2, 1, 5, 4, 11, 11, 22, 26)$ for the unity operator and $(2, 1, 5, 5, 12, 14, 28, 36, 64, 85)$ for the perturbing one; (ii) $n = 4$ $(1, 0, 3, 0, 5, 2, 11, 7, 22, 19)$ for the unity operator and $(1, 2, 3, 5, 9, 13, 22, 33, 52, 77)$ for the perturbing operator; (iii) $n = 5$ $(1, 0, 2, 1, 4, 2, 9, 7, 18, 18)$ for the unity operator and $(1, 1, 4, 3, 9, 10, 21, 26, 48, 63)$ for the perturbing operator. So we see the existence of conserved charges of spin 3 for each n and even a charge of spin 5 for the $n = 5$ case, in full correspondence with the exponents of the B_n ATFT.

As we mentioned in the introduction, the conjectured symmetries of the factorized S-matrices for these integrable field theories are expressed by the algebras which are dual to the corresponding non-simply-laced ATFT (with an imaginary coupling constant), i.e. $D_4^{(3)}$ for G_2 and $A_{2n-1}^{(2)}$ for B_n. In the next section we shall build explicitly non-local charges in the vector perturbed $WD_3^{(p)}$ theory which form the $A_{5,q}^{(2)}$ algebra and shall discuss its representation by fundamental solitons.

3. Algebra of non-local charges and its representation

The construction of non-local charges with $sl(2)_q$ symmetry algebra in Sine–Gordon theory obtained as perturbations of minimal unitary conformal models was done in [20, 21]. There it was shown how this construction may be generalized to any ATFT with another symmetry group, and in [22] the results of Smirnov for the S-matrix of $(1, 2)$ perturbations of minimal models were reproduced from the point of view of non-local charges. In this section we shall follow [22] and construct non-local charges for the integrable model $WD_3^{(p)} + \Phi_{(211|111)}$ which form the algebra $A_{5,q}^{(2)}$. The $D_4^{(3)}$ case is somewhat more special and its detailed consideration with its R- (S-)matrix construction is now under investigation.

So, we are going to consider the perturbed $WD_3^{(p)}$ theory with the Hamiltonian

$$\begin{aligned} H &= \frac{\lambda}{2\pi}\int \mathrm{d}^2z \, \left(\mathrm{e}^{-\mathrm{i}\beta(\Phi_1-\Phi_2)} + \mathrm{e}^{-\mathrm{i}\beta(\Phi_2-\Phi_3)} + \mathrm{e}^{-\mathrm{i}\beta(\Phi_2+\Phi_3)} + \mathrm{e}^{\mathrm{i}\beta\Phi_1}\right) \\ &= \frac{\lambda}{2\pi}\int \mathrm{d}^2z \, \Phi_{\mathrm{pert}}(z, \bar{z}) \end{aligned} \tag{6}$$

where $\Phi_i(z, \bar{z}) = \phi_i(z) + \bar{\phi}_i(\bar{z})$, $i = 1, 2, 3$ are free fields of WD_3, λ is the coupling constant of the perturbation, and $\beta = \sqrt{p/(p+1)}$. We briefly recall the method for constructing non-local conserved charges of a perturbed CFT ([1, 20, 21]). If we assume the existence of some conserved chiral currents $J(z)$, $\bar{J}(\bar{z})$ for a non-perturbed CFT then for the perturbed currents we have the following Zamolodchikov equations up to first order:

$$\begin{aligned}\bar{\partial} J(z,\bar{z}) &= \lambda \oint \frac{\mathrm{d}\omega}{2\pi \mathrm{i}} \Phi_{\mathrm{pert}}(\omega,\bar{z}) J(z)\\ \partial \bar{J}(z,\bar{z}) &= \lambda \oint \frac{\mathrm{d}\bar{\omega}}{2\pi \mathrm{i}} \Phi_{\mathrm{pert}}(z,\bar{\omega}) \bar{J}(\bar{z}) .\end{aligned} \tag{7}$$

If the operator product expansions (OPE) in these contour integrations have the form

$$\begin{aligned}\Phi_{\mathrm{pert}}(\omega,\bar{z}) J(z) &= \frac{\bar{h}(\omega,\bar{z})}{(\omega-z)^2} + \frac{\partial_\omega \bar{f}(\omega,\bar{z})}{(\omega-z)} + \text{regular terms}\\ \Phi_{\mathrm{pert}}(z,\bar{\omega}) \bar{J}(\bar{z}) &= \frac{h(z,\bar{\omega})}{(\bar{\omega}-\bar{z})^2} + \frac{\bar{\partial}_{\bar{\omega}} f(z,\bar{\omega})}{(\bar{\omega}-\bar{z})} + \text{regular terms}\end{aligned} \tag{8}$$

then the Zamolodchikov equations take the form

$$\begin{aligned}\bar{\partial} J(z,\bar{z}) &= \partial \bar{H}(z,\bar{z})\\ \partial \bar{J}(z,\bar{z}) &= \bar{\partial} H(z,\bar{z})\end{aligned} \tag{9}$$

where

$$\bar{H}(z,\bar{z}) = \lambda(\bar{h}(z,\bar{z}) + \bar{f}(z,\bar{z}))$$

$$H(z,\bar{z}) = \lambda(h(z,\bar{z}) + f(z,\bar{z}))$$

which means that the conserved charges

$$\begin{aligned}Q &= \int \frac{\mathrm{d}z}{2\pi \mathrm{i}} J + \int \frac{\mathrm{d}\bar{z}}{2\pi \mathrm{i}} \bar{H}\\ \bar{Q} &= \int \frac{\mathrm{d}z}{2\pi \mathrm{i}} H + \int \frac{\mathrm{d}\bar{z}}{2\pi \mathrm{i}} \bar{J}\end{aligned} \tag{10}$$

must exist. For the case under consideration the 'maximal' set of currents which satisfies equations (8) is

$$\begin{aligned}
&J_1(z) = \mathrm{e}^{(\mathrm{i}/\beta)(\phi_2(z)-\phi_3(z))} && \bar{H}_1(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-1} \mathrm{e}^{\mathrm{i}(\frac{1}{\beta}-\beta)(\phi_2(z)-\phi_3(z))} \mathrm{e}^{-\mathrm{i}\beta(\bar{\phi}_2(\bar{z})-\bar{\phi}_3(\bar{z}))}\\
&J_2(z) = \mathrm{e}^{(\mathrm{i}/\beta)(\phi_1(z)-\phi_2(z))} && \bar{H}_2(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-1} \mathrm{e}^{\mathrm{i}(\frac{1}{\beta}-\beta)(\phi_1(z)-\phi_2(z))} \mathrm{e}^{-\mathrm{i}\beta(\bar{\phi}_1(\bar{z})-\bar{\phi}_2(\bar{z}))}\\
&J_3(z) = \mathrm{e}^{-\mathrm{i}\frac{2}{\beta}\phi_1(z)} && \bar{H}_3(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-2} \mathrm{e}^{-2\mathrm{i}(\frac{1}{\beta}-\frac{\beta}{2})\phi_1(z)} \mathrm{e}^{\mathrm{i}\beta\bar{\phi}_1(\bar{z})}\\
&J_0(z) = \mathrm{e}^{(\mathrm{i}/\beta)(\phi_2(z)+\phi_3(z))} && \bar{H}_0(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-1} \mathrm{e}^{\mathrm{i}(\frac{1}{\beta}-\beta)(\phi_2(z)+\phi_3(z))} \mathrm{e}^{-\mathrm{i}\beta(\bar{\phi}_2(\bar{z})+\bar{\phi}_3(\bar{z}))}
\end{aligned} \tag{11}$$

$$\begin{aligned}
&\bar{J}_1(\bar{z}) = \mathrm{e}^{-(\mathrm{i}/\beta)(\bar{\phi}_2(\bar{z})-\bar{\phi}_3(\bar{z}))} && H_1(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-1} \mathrm{e}^{-\mathrm{i}(\frac{1}{\beta}-\beta)(\bar{\phi}_2(\bar{z})-\bar{\phi}_3(\bar{z}))} \mathrm{e}^{\mathrm{i}\beta(\phi_2(z)-\phi_3(z))}\\
&\bar{J}_2(\bar{z}) = \mathrm{e}^{-(\mathrm{i}/\beta)(\bar{\phi}_1(\bar{z})-\bar{\phi}_2(\bar{z}))} && H_2(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-1} \mathrm{e}^{-\mathrm{i}(\frac{1}{\beta}-\beta)(\bar{\phi}_1(\bar{z})-\bar{\phi}_2(\bar{z}))} \mathrm{e}^{\mathrm{i}\beta(\phi_1(z)-\phi_2(z))}\\
&\bar{J}_3(\bar{z}) = \mathrm{e}^{\mathrm{i}\frac{2}{\beta}\bar{\phi}_1(\bar{z})} && H_3(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-2} \mathrm{e}^{2\mathrm{i}(\frac{1}{\beta}-\frac{\beta}{2})\bar{\phi}_1(\bar{z})} \mathrm{e}^{-\mathrm{i}\beta\phi_1(z)}\\
&\bar{J}_0(\bar{z}) = \mathrm{e}^{-(\mathrm{i}/\beta)(\bar{\phi}_2(\bar{z})+\bar{\phi}_3(\bar{z}))} && H_0(z,\bar{z}) = \lambda \frac{\beta^2}{\beta^2-1} \mathrm{e}^{-\mathrm{i}(\frac{1}{\beta}-\beta)(\bar{\phi}_2(\bar{z})+\bar{\phi}_3(\bar{z}))} \mathrm{e}^{\mathrm{i}\beta(\phi_2(z)+\phi_3(z))}
\end{aligned} \tag{12}$$

with the spins of the charges (10)

$$
\begin{aligned}
s_i &= -\bar{s}_i = \frac{1}{\beta^2} - 1 \qquad i = 0, 1, 2 \\
s_3 &= -\bar{s}_3 = \frac{2}{\beta^2} - 1 \,.
\end{aligned}
\tag{13}
$$

We should recall now that the quasi-chiral components $\phi_i, \bar{\phi}_i$ of the Toda free fields Φ_i commute only in the absence of the perturbation $\lambda = 0$, but in general their vertex operators obey certain braiding relations (see [20]). Omitting details, we will give the result which can easily be checked by the use of these braiding relations: the definition of the topological charges

$$
\begin{aligned}
T_1 &= \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \; \partial_x (\Phi_2 - \Phi_3) \\
T_2 &= \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \; \partial_x (\Phi_1 - \Phi_2) \\
T_3 &= -2 \frac{\beta}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}x \; \partial_x \Phi_1 \\
T_0 &= -2T_2 - T_1 - T_3
\end{aligned}
\tag{14}
$$

leads to the following algebra of charges:

$$
\begin{aligned}
&[T_i, Q_j] = A_{ij} Q_j \\
&[T_i, \bar{Q}_j] = -A_{ij} \bar{Q}_j \\
&Q_i \bar{Q}_j - q^{-A_{ij}} \bar{Q}_j Q_i = \delta_{ij} a_i (1 - q^{2T_i})
\end{aligned}
\tag{15}
$$

where

$$
A_{ij} = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ -1 & -1 & 2 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}
\tag{16}
$$

differs just by diagonal normalization from the Cartan matrix of the affine Lie algebra $A_5^{(2)}$,

$$
q = \mathrm{e}^{-\mathrm{i}\pi/\beta^2}
\tag{17}
$$

is the deformation parameter of quantum group, and

$$
\begin{aligned}
a_i &= \frac{\lambda}{2\pi \mathrm{i}} \left(\frac{\beta^2}{\beta^2 - 1} \right)^2 \qquad i = 0, 1, 2 \\
a_3 &= \frac{\lambda}{2\pi \mathrm{i}} \left(\frac{\beta^2}{\beta^2 - 2} \right)^2 .
\end{aligned}
\tag{18}
$$

The standard substitution ([21, 22])

$$
\begin{aligned}
Q_i &= c_i \mathrm{e}^{s_i \theta} E_i q^{H_i/2} \\
\bar{Q}_i &= c_i \mathrm{e}^{\bar{s}_i \theta} F_i q^{H_i/2} \\
H_i &= T_i \\
c_i^2 &= a_i (q_i^{-2} - 1) \\
q_i &= q^{A_{ii}/2}
\end{aligned}
\tag{19}
$$

which introduces the spectral parameter (θ) dependence, transforms the algebra (15) into the quantum affine Lie algebra $A_{5q}^{(2)}$ with the Chevalley basis E_i, F_i, H_i ($i = 0, 1, 2, 3$):

$$\begin{aligned}
&[H_i, H_j] = 0\\
&[H_i, E_j] = A_{ij}E_j\\
&[H_i, F_j] = -A_{ij}F_j\\
&[E_i, F_j] = \delta_{ij}\frac{q^{H_i} - q^{-H_i}}{q_i - q_i^{-1}}\,.
\end{aligned}\tag{20}$$

The fundamental representation of this algebra can be chosen the same as for the $A_5^{(2)}$ and the basis for Cartan subalgebra in this representation can be taken as $H_1 = \mathrm{diag}(1, 0, 0, 0, 0, -1)$; $H_2 = \mathrm{diag}(0, 1, 0, 0, -1, 0)$; $H_3 = \mathrm{diag}(0, 0, 1, -1, 0, 0)$; $H_0 = -2H_2 - H_1 - H_3$.

The next natural step is the construction of the sextet of fundamental soliton fields for the model which will form the representation of the above written algebra of non-local currents. One of possible candidates can be chosen as $\psi_{i\pm} = e^{\pm(i/\beta)\phi_i}$, where $i = 1, 2, 3$, or another choice : $\bar{\psi}_{i\pm} = e^{\pm(i/\beta)\bar{\phi}_i}$. It can easily be checked by the standard technique of conformal field theory that any of these two sets of fields possesses the correct topological charges, namely

$$\begin{array}{lll}
[T_0, \psi_{1\pm}] = 0 & [T_0, \psi_{2\pm}] = \pm\psi_{2\pm} & [T_0, \psi_{3\pm}] = \pm\psi_{3\pm}\\
[T_1, \psi_{1\pm}] = 0 & [T_1, \psi_{2\pm}] = \pm\psi_{2\pm} & [T_1, \psi_{3\pm}] = \mp\psi_{3\pm}\\
[T_2, \psi_{1\pm}] = \pm\psi_{1\pm} & [T_2, \psi_{2\pm}] = \mp\psi_{2\pm} & [T_2, \psi_{3\pm}] = 0\\
[T_3, \psi_{1\pm}] = \mp 2\psi_{1\pm} & [T_3, \psi_{2\pm}] = 0 & [T_3, \psi_{3\pm}] = 0\,.
\end{array}\tag{21}$$

Clearly each of the set of fields ψ, $\bar{\psi}$ suffers from the ill-defined action of the part of the charges Q_i, $\bar{Q}_i$ because of the branch cuts under the contour integrals in part of the operator product expansions $Q\psi$, and hence does not form the correct representation of the full algebra (15). But actually for our purposes here we need only the fields which will permit us to define braiding relations between currents and fundamental soliton fields which are compatible with the comultiplication structure of the revealed group $A_5^{(2)}$. Using relations (21) one can show by the technique of the braiding relations of the vertices for fields ϕ [20] that for the fields ψ defined above the following braiding relations are valid:

$$\begin{aligned}
J_i(x)\psi_{j\pm}(y) &= q^{\tau_{ij\pm}}\psi_{j\pm}(y)J_i(x)\\
\bar{J}_i(x)\psi_{j\pm}(y) &= q^{-\tau_{ij\pm}}\psi_{j\pm}(y)\bar{J}_i(x)
\end{aligned}\tag{22}$$

for $x < y$, and commutation of $J_i(x)$ and $\psi_{j\pm}(y)$ for $x > y$, where the $\tau_{ij\pm}$ are the topological charges of the fields $\psi_{j\pm}$ with respect to T_i, and the latter can be read from (21). Such braiding relations induce the comultiplication

$$\begin{aligned}
\Delta(Q_i) &= Q_i \otimes 1 + q^{H_i} \otimes Q_i\\
\Delta(\bar{Q}_i) &= \bar{Q}_i \otimes 1 + q^{H_i} \otimes \bar{Q}_i\\
\Delta(H_i) &= H_i \otimes 1 + 1 \otimes H_i
\end{aligned}\tag{23}$$

which acts on the tensor product of two-soliton states.

4. S-matrix construction

The S-matrix should commute with the comultiplication (23):

$$[S, \Delta(H_i)] = [S, \Delta(Q_i)] = [S, \Delta(\bar{Q}_i)] = 0\,.\tag{24}$$

Introducing the notation $\hat{S} = PS$, where P is the permutation matrix, these equations can be rewritten as

$$\begin{aligned}
&\left[\hat{S}(\theta_1, \theta_2), \Delta(H_i)\right] = 0 \\
&\hat{S}(\theta_1, \theta_2)\left(e_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes e_i\right) = \left(q^{-H_i/2} \otimes e_i + e_i \otimes q^{H_i/2}\right)\hat{S}(\theta_1, \theta_2) \\
&\hat{S}(\theta_1, \theta_2)\left(f_i \otimes q^{-H_i/2} + q^{H_i/2} \otimes f_i\right) = \left(q^{-H_i/2} \otimes f_i + f_i \otimes q^{H_i/2}\right)\hat{S}(\theta_1, \theta_2)
\end{aligned} \tag{25}$$

where $\theta_{1,2}$ are the rapidities of the incoming particles,

$$e_i = x_i E_i \qquad f_i = x_i^{-1} F_i \qquad x_i = x_i(\theta_j) = e^{s_i \theta_j} \tag{26}$$

and the dependence of e_i and f_i on the rapidity (θ_1 or θ_2) is defined by their positions in the tensor product (they depend on θ_1 in the first place, and on θ_2 in the second).

A system of equations like (25) was solved with respect to S (without unitarity and crossing symmetry conditions) by Jimbo [19] for almost all affine algebras of symmetry of the R-matrix S in the vector representation (except for $E_6^{(1)}$, $E_7^{(2)}$, $E_8^{(1)}$, $G_2^{(1)}$, $F_4^{(1)}$, $E_6^{(2)}$, $D_4^{(3)}$ cases), but he used another Cartan basis and his results were obtained in other gradation. His Cartan basis h_i is connected with our basis H_i by the transformation

$$h_1 = H_1 - H_2 \qquad h_2 = H_2 - H_3 \qquad h_3 = H_2 + H_3 \qquad h_0 = -2H_1 \tag{27}$$

with a corresponding change in the Chevalley generators E_i', F_i'. The R-matrix was obtained in a homogeneous gradation, where the spectral parameter dependence x is introduced as a multiplier of only one of the Chevalley generators, which corresponds to the highest-weight of A_5 in its odd component of the Dynkin diagram automorphism decomposition—the root number 3 in our case. So in this homogeneous gradation

$$\begin{aligned}
&e_3^{\text{hom}} = x E_3' \qquad f_3^{\text{hom}} = x^{-1} F_3' \qquad x = x(\theta_2 - \theta_1) \\
&e_i^{\text{hom}} = E_i' \qquad f_i^{\text{hom}} = F_i' \qquad i = 0, 1, 2
\end{aligned} \tag{28}$$

and the R-matrix S is a function of x. If we want to make use of Jimbo's result for the $A_5^{(2)}$ R-matrix in our S-matrix construction, we should change the homogeneous gradation into spin, (which, as we saw, was naturally dictated by the non-local charges of the system) by 'gauge' transformation of Jimbo's solution:

$$\tilde{R}(x, k) = \sigma_{21} R(x, k) \sigma_{12}^{-1} \tag{29}$$

where

$$\sigma_{12} = x_0(\theta_1)^{-\sum_{i=1}^{3} h_i a_i/2} \otimes x_0(\theta_2)^{-\sum_{i=1}^{3} h_i a_i/2}. \tag{30}$$

Using the obvious relations

$$\begin{aligned}
&y^{h_i/2} E_j' y^{-h_i/2} = y^{a_{ij}/2} E_j' \\
&y^{h_i/2} F_j' y^{-h_i/2} = y^{-a_{ij}/2} F_j'
\end{aligned} \tag{31}$$

we have the following system of equations which fixes a_i in (30):

$$\begin{aligned}
&x_0{}^{a_1 - a_2/2 - a_3/2} = x_0 \\
&x_0{}^{-a_1/2 + a_2} = x_0 \\
&x_0{}^{-a_1/2 + a_3} = x_0 \\
&x_0{}^{-a_1} x = x_1 .
\end{aligned} \tag{32}$$

Solving the first three equations, we have $a_1 = 4$, $a_2 = a_3 = 3$, and the last equation gives us the important relation

$$x = x_1 x_0^4 . \tag{33}$$

Since the dependence on the coupling constant β enters Jimbo's R-matrix only through its dependence on x, relation (33) gives us the effective coupling constant ξ as a function of β:

$$x = e^{2\pi\theta/\xi} \qquad \frac{2\pi}{\xi} = \frac{6}{\beta^2} - 5\,. \tag{34}$$

Here we will write down the R-matrix with the $A_{2n-1}^{(2)}$ symmetry group in the homogeneous gradation as obtained by Jimbo in [19] and will show its crossing symmetry transformation. It has the following form:

$$\begin{aligned} R(x,k) = {} & (x-k^2)(x+k^{2n})\sum_{\alpha\neq\alpha'} E_{\alpha\alpha}\otimes E_{\alpha\alpha} + k(x-1)(x+k^{2n})\sum_{\alpha\neq\beta,\beta'} E_{\alpha\alpha}\otimes E_{\beta\beta} \\ & -(k^2-1)(x+k^{2n})\left(\sum_{\alpha<\beta,\alpha\neq\beta'} + x\sum_{\alpha>\beta,\alpha\neq\beta'}\right) E_{\alpha\beta}\otimes E_{\beta\alpha} \\ & + \sum a_{\alpha\beta}(x) E_{\alpha\beta}\otimes E_{\alpha'\beta'} \end{aligned} \tag{35}$$

where

$$a_{\alpha\beta}(x) = \begin{cases} (1-k^2)(k^{2n+\bar\alpha-\bar\beta}(x-1) + \delta_{\alpha\beta'}(x+k^{2n})) & \alpha<\beta \\ (k^2-1)x(k^{\bar\alpha-\bar\beta}(x-1) - \delta_{\alpha\beta'}(x+k^{2n})) & \alpha>\beta \\ k(x+k^{2n})(x-1) - (k^{2n}+1)(k^2-1)x & \alpha=\beta=\alpha' \\ k^2x + k^{2n}(x-1) & \alpha=\beta, \alpha\neq\alpha' \end{cases}$$
$$\bar\alpha = \begin{cases} \alpha+\frac{1}{2} & 1\leqslant\alpha\leqslant n \\ \alpha-\frac{1}{2} & n+1\leqslant\alpha\leqslant 2n \end{cases} \tag{36}$$

and the matrices $E_{\alpha\beta} = \delta_{i\alpha}\delta_{j\beta}$ with indices $\alpha = 1,\ldots,2n$ and $\alpha' = 2n+1-\alpha$.

One can check that the R-matrix (35) has the following crossing transformation property: under the change

$$\begin{aligned} & x \to -\frac{k^{2n}}{x} \\ & E_{\alpha\beta}\otimes E_{\gamma\delta} \to k^{\bar\alpha'-\bar\beta'} E_{\gamma\delta}\otimes E_{\beta'\alpha'} \end{aligned} \tag{37}$$

the quantity $\hat R = x^{-1}k^{-n}R$ remains unchanged.

Another sufficient property of (35) which one can check is its unitarity condition:

$$\hat R(x,k)\hat R(x^{-1},k) = (x^{\frac12}k^{-1} - x^{-\frac12}k)(x^{-\frac12}k^{-1} - x^{\frac12}k)(x^{\frac12}k^{-n} + x^{-\frac12}k^{n})(x^{-\frac12}k^{-n} + x^{\frac12}k^{n}). \tag{38}$$

One can now easily check for $n=3$ using (37), (29), (30) with the previously found values of a_i that the 'gauged' R-matrix $\tilde R(x,k) = \sigma_{21}\hat R(x,k)\sigma_{12}^{-1}$ becomes crossing invariant, since the factor $k^{\bar\alpha'-\bar\beta'}$ is now exactly compensated by 'σ'-factors. Using this fact one can see that if we now look for the S-matrix in the form

$$S(x,k) = S_0(x,k)\sigma_{21}\hat R(x,k)\sigma_{12}^{-1} \tag{39}$$

where S_0 is some unitarizing factor, then it should obey the following system of crossing and unitarity conditions:

$$S_0(-k^6x^{-1},k) = S_0(x,k) \tag{40}$$

$$S_0(x,k)S_0(x^{-1},k) = [(x^{\frac{1}{2}}k^{-1} - x^{-\frac{1}{2}}k)(x^{-\frac{1}{2}}k^{-1} - x^{\frac{1}{2}}k)]^{-1}$$
$$\times[(x^{\frac{1}{2}}k^{-n} + x^{-\frac{1}{2}}k^{n})(x^{-\frac{1}{2}}k^{-n} + x^{\frac{1}{2}}k^{n})]^{-1}. \tag{41}$$

Before we solve these equations, we would now like to fit the deformation parameter k with the coupling constant by use of the crossing transformation for the R-matrix solution ($x \to -x/k^6$ corresponding to $\theta \to \mathrm{i}\pi - \theta$) then in the notation

$$x = \mathrm{e}^{-\mathrm{i}\pi a} \qquad a = \frac{2\mathrm{i}\theta}{\xi} \qquad k = \mathrm{e}^{-\mathrm{i}\pi b} \tag{42}$$

we get

$$b = -\frac{\pi}{3\xi} + \frac{1}{6}. \tag{43}$$

We have the usual crossing condition for S_0: $S_0(\theta) = S_0(\mathrm{i}\pi - \theta)$, and the unitarity condition (41) in the form

$$S_0(\theta)S_0(-\theta) = \frac{1}{\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)\operatorname{sh}\frac{\pi}{\xi}(\theta + \mathrm{i}\pi)\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\gamma)\operatorname{sh}\frac{\pi}{\xi}(\theta + \mathrm{i}\gamma)} \tag{44}$$

with

$$\gamma = \frac{1}{3}\left(\pi - \frac{\xi}{2}\right). \tag{45}$$

Among the infinite number of solutions of (44) together with the crossing symmetry condition, one can choose the following one with the minimal number of poles on the physical strip:

$$S_0(\theta) = \pm\frac{1}{\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\gamma)}\prod_{l=0}^{\infty}\frac{\operatorname{sh}\frac{\pi}{\xi}(\theta + \mathrm{i}\pi 2l)\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\pi(2l+1))}{\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\pi 2l)\operatorname{sh}\frac{\pi}{\xi}(\theta + \mathrm{i}\pi(2l+1))}$$
$$\times\frac{\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\gamma + \mathrm{i}\pi(2l+1))\operatorname{sh}\frac{\pi}{\xi}(\theta + \mathrm{i}\gamma - \mathrm{i}\pi 2(l+1))}{\operatorname{sh}\frac{\pi}{\xi}(\theta + \mathrm{i}\gamma - \mathrm{i}\pi(2l+1))\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\gamma + \mathrm{i}\pi 2(l+1))} \tag{46}$$

$$= \pm\frac{1}{\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\pi)\operatorname{sh}\frac{\pi}{\xi}(\theta - \mathrm{i}\gamma)}\exp\left(-\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}k}{k}\frac{\sin k\theta\operatorname{sh}\frac{k\gamma}{2}}{\operatorname{sh}\frac{k\xi}{2}\operatorname{ch}\frac{\pi k}{2}}\operatorname{ch}\frac{\xi + \gamma - \pi}{2}k\right). \tag{47}$$

(The connection between (46) and (47) is explained in the appendix).

In principle the solution of crossing and unitarity conditions presented above could be enough for our goal, namely the construction of the S-matrix for the restricted model. In the above consideration coupling constant parameter β was arbitrary, but as is well known, the special and crucial property of minimal model perturbations is the rationality of β^2, or in other words, the property of q to be a root of unity, namely

$$\beta = \sqrt{\frac{p}{p+1}} \qquad \text{or} \qquad \beta = -\sqrt{\frac{p+1}{p}}. \tag{48}$$

The first choice in (48) is relevant for the perturbation under consideration, whereas the second corresponds to another integrable perturbation of the same model, $(11\ldots 1|21\ldots 1)$, which may be considered as a generalization of $(2,1)$ perturbation of minimal Virasoro models. As is well known the kink–kink S-matrix construction in the framework of the scheme presented needs the explicit kink–soliton correspondence including the Clebsh–Gordan coefficients at the root of unity for the tensor product of the quantum group vector representations. Such a correspondence is equivalent to the change of basis and effectively means [8] the construction of IRF model based on the weight lattice of the same symmetry group ($A_5^{(2)}$ in our case). Fortunately, such IRF models with the proper symmetry group have been constructed in [9, 10] and we will use these results with some modifications explained below for our fundamental S-matrix construction.

Let us note at this point that the effective coupling constant ξ, equation (34), as a function of k takes the following form when the first choice in (48) is taken:

$$\xi = \frac{\pi(8+2k)}{10+k}\,. \tag{49}$$

We will now write down the Boltzmann weights of the $A_{2n-1}^{(2)}$ RSOS model based on the realization for this algebra made by using the D_n loop algebra [10] (in contrast to the C_n realized $A_{2n-1}^{(2)}$ Boltzmann weights constructed in [9]). We will write them for the restricted model in the trigonometric limit and will use the notation of [9] and not of the paper [10] itself. (The change of parameters which transforms the Boltsmann weights of [10] into those of [9] for earlier known solutions is written in [10]).

Let us fix some notation. Λ_i $(0 \leqslant i \leqslant n)$ denote the fundamental weights of $D_n^{(1)}$, and $\rho = \Lambda_0 + \cdots + \Lambda_n$. Let $\mathcal{A}$ be the set of weights in the vector representation of D_n and for $a \in \mathcal{H}^* \equiv \sum_{i=0}^{n} \mathbb{C}\Lambda_i$ we write $\bar{a}$ to mean its classical part. In terms of the orthogonal vectors e_i $(1 \leqslant i \leqslant n)$, $(e_i, e_j) = \delta_i j$, $e_{-i} = -e_i$, the classical parts $\bar{\Lambda}_i, \bar{\rho}$ and $\mathcal{A}$ can be written as follows:

$$\begin{aligned} &\bar{\Lambda}_i = e_1 + \cdots + e_i \qquad (1 \leqslant i \leqslant n-2) \\ &\bar{\Lambda}_{n-1} = \tfrac{1}{2}(e_1 + \cdots + e_{n-1} - e_n) \\ &\bar{\Lambda}_n = \tfrac{1}{2}(e_1 + \cdots + e_{n-1} + e_n) \end{aligned} \tag{50}$$

$$a = (L - a_1 - a_2 - 1)\Lambda_0 + \sum_{i=1}^{n-1}(a_i - a_{i+1} - 1)\Lambda_i + (a_{n-1} + a_n - 1)\Lambda_n$$

$$L > a_1 + a_2 \quad a_1 > a_2 > \cdots > a_n \quad a_{n-1} + a_n > 0 \tag{51}$$

where $a_i \in \mathbb{Z}$ or $a_i \in \mathbb{Z} + \frac{1}{2}$ and $L = 2n - 2 + k$, $(k = 1, 2, \ldots)$ is the number of minimal unitary WD model which we perturb. Let us point out here that the half-integer sector of the theory is irrelevant in our case of vector perturbations of CFT without fermions, and only the integer sector will be relevant for our S-matrix construction. It can be easily seen that

$$\bar{a} + \bar{\rho} = \sum_{i=1}^{n} a_i e_i \qquad a_\mu = \langle a + \rho, e_\mu \rangle \quad -n \leqslant \mu \leqslant n. \tag{52}$$

It was shown in [10] that the Boltzmann weights (here we write them in the trigonometric limit, while in [10] they are written in general elliptic form) for this RSOS model take the form

$$[x] = \sin \omega x \qquad [x]_+ = \cos \omega x \qquad \omega = \frac{\pi}{L_s}$$

$$W_u \begin{pmatrix} a & a+e_\mu \\ a+e_\mu & a+2e_\mu \end{pmatrix} = \frac{[1+u][n+u]_+}{[1][n]_+} \qquad (\mu \neq 0)$$

$$W_u \begin{pmatrix} a & a+e_\mu \\ a+e_\mu & a+e_\mu+e_\nu \end{pmatrix} = \frac{[a_{\mu\nu} - u][n+u]_+}{[a_{\mu\nu}][n]_+} \qquad (\mu \neq \pm\nu)$$

$$W_u\begin{pmatrix} a & a+e_\nu \\ a+e_\mu & a+e_\mu+e_\nu \end{pmatrix} = \left(\frac{[a_{\mu\nu}+1][a_{\mu\nu}-1]}{[a_{\mu\nu}]^2}\right)^{1/2}\frac{[u][n+u]_+}{[1][n]_+} \qquad (\mu\neq\pm\nu)$$

$$W_u\begin{pmatrix} a & a+e_\nu \\ a+e_\mu & a \end{pmatrix} = \left(G_{a,\mu}G_{a,\nu}\right)^{1/2}\frac{[u][a_{\mu-\nu}+1-n-u]_+}{[a_{\mu-\nu}+1][n]_+} \qquad (\mu\neq\nu)$$

$$W_u\begin{pmatrix} a & a+e_\mu \\ a+e_\mu & a \end{pmatrix} = \frac{[2a_\mu+1-u][n+u]_+}{[2a_\mu+1][n]_+} + \frac{[u][2a_\mu+1-n-u]_+}{[2a_\mu+1][n]_+}G_{a,\mu} \qquad (\mu\neq 0)$$

$$= \frac{[2a_\mu+1-2n-u][n-u]_+}{[2a_\mu+1-2n][n]_+} - \frac{[u][2a_\mu+1-n-u]_+}{[2a_\mu+1-2n][n]_+}H_{a,\mu} \tag{53}$$

where $a_{\mu\nu} = a_\mu - a_\nu$, $a_{\mu-\nu} = a_\mu + a_\nu$

$$G_{a,\mu} = G_{a+e_\mu}/G_a = \begin{cases} \prod_{k\neq 0,\pm\mu}\frac{[a_{\mu k}+1]}{[a_{\mu k}]} & \mu\neq 0 \\ 1 & \mu = 0 \end{cases}$$

$$G_a = \prod_{1\leqslant i<j\leqslant n}[a_i-a_j][a_i+a_j] \tag{54}$$

$$H_{a,\mu} = \sum_{k\neq\mu}\frac{[a_\mu+a_k+1-2n]}{[a_\mu+a_k+1]}G_{a,k}$$

and $L_s = L/s$, where s is in principle any integer coprime with L.

The unitarity and crossing relations for these Boltzmann weights read as

$$\sum_g W_u\begin{pmatrix} a & g \\ c & d \end{pmatrix} W_{-u}\begin{pmatrix} a & b \\ g & d \end{pmatrix} = \delta_{bc}\frac{[n-u]_+[n+u]_+[1+u][1-u]}{[n]_+^2[1]^2} = \delta_{bc}\rho(u) \tag{55}$$

$$W_u\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{G_bG_c}{G_aG_d}\right)^{1/2} W_{k/2-1-u}\begin{pmatrix} c & a \\ d & b \end{pmatrix}. \tag{56}$$

We now look for the S-matrix of the scattering process of kinks in the form [11, 13]

$$S_u\begin{pmatrix} a & b \\ c & d \end{pmatrix} = Y(u)W_{\eta u}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\left(\frac{G_aG_d}{G_bG_c}\right)^{u/2} \tag{57}$$

with some scalar function Y to be found, where u is connected with the rapidity difference of the incoming kinks θ by $u = \theta/\pi\mathrm{i}$, and η is some constant.

Before the constraints on Y are derived let us fix the parameter s for L_s. Explicit comparison of the unitarity condition for Boltzmann weights (55) with the unitarity relation for the R-matrix (44) with the restricted model relation (49) taken into account tells us that the correct choice is $s = -1$. With this choice the unitarity constraint can be satisfied by virtue of relation (55) provided

$$Y(u)Y(-u) = 1/\rho(\eta u) \tag{58}$$

while the crossing relation is satisfied provided η is equal to the crossing parameter, which with $s = -1$ is equal to $\eta = -3 - (4+k)/2$, and

$$Y(u) = Y(1-u). \tag{59}$$

The system of functional equations (58) and (59) exactly coincide with the analogous system for S_0 in the previous section, which we have already solved. So we conjecture the solution $Y = S_0$. As is well known, the solution to the crossing and unitarity conditions has an ambiguity in the form of the arbitrary product of CDD factors. In many physical cases this ambiguity is solved by the 'principle of minimality', which supposes the minimal number of poles for the fundamental S-matrix in the physical strip. Since we expect this principle to work in our case, we do not add any CDD factor to the minimal solution. (Any such CDD factor necessarily has a pole in the physical strip). As will be checked below in particular cases when the S-matrix is known, this leads to the correct answer.

One can see that the S-matrix of the vector perturbed $WD_n^{(k)}$ minimal theories for $k = 1$ should take the form of the (non-restricted) Sine–Gordon S-matrix at a special value of its coupling constant, since the central charge for this k is equal to 1 for each n. Moreover, since the dimension of the perturbation for $k = 1$, eqaution (5), is the inverse of an even number, we should expect to have some subsector of SG S-matrix at the reflectionless point. Indeed, in the case $k = 1$ and, for example, $n = 3$ the restriction condition (51) leaves only four possibilities for the choice of (a_3, a_2, a_1): $(3, 1, 0)$; $(2, 1, 0)$; $(\frac{5}{2}, \frac{3}{2}, \frac{1}{2})$ and $(\frac{5}{2}, \frac{3}{2}, -\frac{1}{2})$. The first set of a_i corresponds to the highest weight of the vector (v) representation of D_3, the second to that of the scalar (0) representation, and the fourth and third to those of the two spinor representations, which are irrelevant for our case. (Only the integer sublattice of weights is relevant for D-type vector perturbations). Using the admissibility condition, we have the following two non-zero Boltzmann weights in this case:

$$W_u \begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \qquad W_u \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} . \tag{60}$$

Both of them are representatives of the last type of non-zero Boltzmann weights in (53) and explicit calculation gives the same expression for each of them:

$$W(u) = \frac{[3-u]_+[1+u]}{[1][3]_+} .$$

For this case the crossing factor $(G_a G_d / G_b G_c)^{u/2}$ turns out to be equal to 1 for both types of S-matrix. It can easily be shown that the tensor structure of the Boltzmann weights (53) is such that at the points $[n+u]_+ = 0$ and $[1-u] = 0$ it becomes a projector onto the scalar and adjoint representations of D_n, respectively. So we see that in the $k = 1$ case contraction of the zero of W and the pole of prefactor F corresponding to the adjoint representation takes place, and the S-matrix becomes a scalar representation projector.

This can be explained in terms of Sine–Gordon (SG) theory if we interpret our fundamental particle as a lightest breather of the SG. The obtained physical strip poles and zeros of the S-matrix exactly coincide with those of lightest breather-breather S-matrix in SG theory [24]

$$S(\theta) = \frac{\operatorname{sh}\theta + \mathrm{i}\sin\frac{\pi}{11}}{\operatorname{sh}\theta - \mathrm{i}\sin\frac{\pi}{11}} \tag{61}$$

at the reflectionless point $\beta^2/8\pi = \frac{1}{12}$ (or in other words $\xi = \frac{\pi}{11}$), in full correspondence with the conformal dimension of the perturbing operator. It means that these two S-matrices should coincide.

Another interesting limit of the S-matrix obtained is the limit $k \to \infty$, in which the model under consideration takes the form of free fermions, the $SO(2n)_1$ Kac–Moody perturbed by the field of conformal dimension $\frac{1}{2}$ (see equation (5)). Therefore we expect the trivial limit (-1) for the S-matrix. It can be checked that the proposed solution for S-matrix has this property. First of all, as was pointed out in [11], for the parameters a_μ this limit means that $a_\mu, a_{\mu\nu} \to \infty$, $a_\mu/k, a_{\mu\nu}/k \to 0$. One can show that the infinite product in Y goes to 1 in the limit $k \to \infty$. It can easily be checked that the prefactor before the infinite product in Y together with the five types of Boltzmann weights (53) gives the zero limit for all of them except for the first and the third one, for which the limit is equal to -1, giving the ± 1 limit for the S-matrix. From the $k \to \infty$ point of view, any CDD combination with the property $\alpha_i(k) \to 0$ will not spoil this property of the S-matrix, but as we said above such CDD factors should be rejected by low-k checking arguments.

Of course, it would be plausible to have a justification that the proposed S-matrix is correct by using other methods. One of them is the TBA check of the ground-state energy in the ultraviolet limit and its comparison with the central charge of the perturbed conformal model. Such a check should confirm the correctness of the S-matrix answer for all values of k.

5. Summary

We have shown by counting of arguments that the $(21\ldots1|11\ldots1)$-perturbations of WD_n theories (and $(21|11)$ of W_3) are integrable, which was not surprizing after we realized their B_n (and G_2) imaginary coupled ATFT structure at the 'classical level'. The conjectured $A^{(2)}_{2n-1}$ symmetry of their quantum S-matrix has been proved by explicit construction of non-local charges with this quantum group symmetry and it was shown that there are a set of fields, which play the role of fundamental solitons in the sense that their braiding relations with non-local charges give rise to the correct comultiplication for this quantum group. As was shown above, the choice of the correct gradation (spin gradation in our case) for the R-matrix of corresponding symmetry has a crucial role for the crossing symmetry of the S-matrix, and, moreover, allows us to find the effective coupling constant for the model. The gradation role problem in S-matrix construction was recently considered in general in [26] with the example of the $C^{(1)}_2$ S-matrix. As was mentioned there, the mass ratios of the particles corresponding to the S-matrix poles depend on the choice of gradation and the spin gradation turns out to be compatible with the duality conjecture for the imaginary coupled affine Toda theories, which says that the mass spectrum of the Toda with affine group G should be described by the S-matrix of the dual group of symmetry $\tilde{G}$. As was pointed out in [7], our result justifies this conjecture too: the pole of the fundamental S-matrix corresponding to the scalar representation reproduces the mass of lightest particle of the $B^{(1)}_3$ real coupled ATFT.

We expect the mass spectrum, particle content (higher kinks and breathers) and structure of the full S-matrix to be rather complicated for the general case, since even in the Virasoro analogue of our model—the $(1, 2)$ perturbation of minimal models—the full spectrum of particles was not obtained in the general case of an arbitrary minimal model. In the same way as was shown in [9], one can show that the problem of spectral decomposition for our R-matrix is equivalent to this problem for two arbitrary representations obtained by the tensor product of the vector representation of the D_n algebra, which is unknown in the general case. Some examples of bootstrap for the $SO(n)$ symmetric R-matrix in the simplest particular cases have been shown in [25] and led to a complicated picture.

The examples of integrable perturbations of W-invariant theories analysed in this paper certainly do not exhaust all of them, which reveal richer structure than in the Virasoro case. It can be seen by comparison of the Dynkin diagrams of affine Lie algebras with those of non-affine Lie algebras of other type X. Each case, where the former can be obtained from the latter X by adding to X some combination of its fundamental weights, might be considered as a candidate for integrability of the perturbation of WX by the field corresponding to this specific combination of the weights of X. For example, it was recently conjectured in [27] that vector perturbed WB_n minimal models are integrable and that their S-matrices have $A_{2n}^{(2)}$ symmetry, which was expressed in terms of the B_n RSOS model with special modification of its crossing parameter†. However, each of these cases require separate detailed investigation.

Appendix

Here we will show how the exponential representation for the infinite product (46) can be obtained. Let us denote it by Σ_0. Then

$$\mathrm{i}\ln\Sigma_0 = \sum_{l=0}^{\infty}\left[-\mathrm{i}\ln\frac{\mathrm{sh}\,\frac{\pi}{\xi}(\theta-\mathrm{i}\pi 2l)}{\mathrm{sh}\,\frac{\pi}{\xi}(\theta+\mathrm{i}\pi 2l)} + \mathrm{i}\ln\frac{\mathrm{sh}\,\frac{\pi}{\xi}(\theta-\mathrm{i}\gamma+\mathrm{i}\pi(2l+1))}{\mathrm{sh}\,\frac{\pi}{\xi}(\theta+\mathrm{i}\gamma-\mathrm{i}\pi(2l+1))}\right.$$
$$\left. + \mathrm{i}ln\frac{\mathrm{sh}\,\frac{\pi}{\xi}(\theta-\mathrm{i}\pi(2l+1))}{\mathrm{sh}\,\frac{\pi}{\xi}(\theta+\mathrm{i}\pi(2l+1))} - \mathrm{i}\ln\frac{\mathrm{sh}\,\frac{\pi}{\xi}(\theta-\mathrm{i}\gamma+\mathrm{i}\pi 2(l+1))}{\mathrm{sh}\,\frac{\pi}{\xi}(\theta+\mathrm{i}\gamma-\mathrm{i}\pi 2(l+1))}\right]. \tag{A1}$$

Suppose that θ is purely imaginary. Then one can use the following integral representation for the function:

$$\mathrm{i}\ln\frac{\mathrm{sh}\,\frac{\pi}{\xi}(\theta+\mathrm{i}\alpha)}{\mathrm{sh}\,\frac{\pi}{\xi}(\theta-\mathrm{i}\alpha)} = \text{constant} - 2\int_C\frac{\mathrm{d}k}{k}\sin(k\mathrm{i}\theta)\frac{\sin(\frac{\xi}{2}-\alpha)k}{\sin\frac{\xi k}{2}} \tag{A2}$$

which is valid for any real ξ and α when θ is purely imaginary, and the contour C goes from 0 to $+\infty$ along and above the real k-axis. The constant depends on the Reimann sheet for the logarithm but cancels when (A2) is inserted into (A1). One can see that

$$\mathrm{i}\ln\Sigma_0 = -2\int_C\frac{\mathrm{d}k}{k}\sin(k\mathrm{i}\theta)\frac{\sigma(k)}{\sin\frac{k\xi}{2}\cos\frac{\pi k}{2}} \tag{A3}$$

with

$$\sigma(k) = \sum_{l=0}^{\infty}\cos\frac{\pi k}{2}\left[-\sin\left(\frac{\xi}{2}+2\pi l\right)k + \sin\left(\frac{\xi}{2}+\gamma-\pi(2l+1)\right)k\right.$$
$$\left. + \sin\left(\frac{\xi}{2}+\pi(2l+1)\right)k - \sin\left(\frac{\xi}{2}+\gamma-2\pi(l+1)\right)k\right]$$
$$= 2\cos\frac{\xi+\gamma-\pi}{2}k\sum_{l=0}^{\infty}\cos\frac{\pi k}{2}\left[\sin\left(2\pi l+\frac{3\pi}{2}-\frac{\gamma}{2}\right)k\right.$$
$$\left. - \sin\left(2\pi l+\frac{\pi}{2}-\frac{\gamma}{2}\right)k\right]. \tag{A4}$$

One can now easily check that

$$\sigma(k) = \cos\frac{\xi+\gamma-\pi}{2}k\sin\frac{k\gamma}{2}. \tag{A5}$$

We see that equation (A3) with (A5) reproduces the equivalent representation (47) for the infinite product, if we rotate the integration contour C to the positive part of the imaginary

† Let us point out that the S-matrix for vector perturbed WD_n minimal models, which was also constructed in this work and is based on the known D-type RSOS models with modified crossing parameter, seems to have the same physical strip pole structure as our's.

k-axis, which may be done for $|\operatorname{Im}\theta|+|\gamma|/2+|\xi+\gamma-\pi|/2<(|\xi|+\pi)/2$. Let us point out here that this condition is satisfied for the physical strip ($0 \leqslant \operatorname{Im}\theta \leqslant \pi$) for minimal models, when ξ, γ, β get special values dictated by (48).

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